

# A COMPACTNESS RESULT FOR A GELFAND-LIOUVILLE SYSTEM WITH LIPSCHITZ CONDITION.

SAMY SKANDER BAHOURA

**ABSTRACT.** We give a quantization analysis to an elliptic system (Gelfand-Liouville type system) with Dirichlet condition. An application, we have a compactness result for an elliptic system with Lipschitz condition.

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## 1. INTRODUCTION AND MAIN RESULTS

We set  $\Delta = \partial_{11} + \partial_{22}$  on open set  $\Omega$  of  $\mathbb{R}^2$  with a smooth boundary.

We consider the following equation:

$$(P) \begin{cases} -\Delta u = V e^v & \text{in } \Omega \subset \mathbb{R}^2, \\ -\Delta v = W e^u & \text{in } \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{in } \partial\Omega, \\ v = 0 & \text{in } \partial\Omega. \end{cases}$$

Here:

$$0 \in \partial\Omega$$

When  $u = v$ , the above system is reduced to an equation which was studied by many authors, with or without the boundary condition, also for Riemann surfaces, see [1-16], one can find some existence and compactness results, also for a system.

Among other results, we can see in [6] the following important Theorem,

**Theorem A.** (Brezis-Merle [6]). *Consider the case of one equation; if  $(u_i)_i = (v_i)_i$  and  $(V_i)_i = (W_i)_i$  are two sequences of functions relatively to the problem (P) with,  $0 < a \leq V_i \leq b < +\infty$ , then, for all compact set  $K$  of  $\Omega$ ,*

$$\sup_K u_i \leq c = c(a, b, K, \Omega).$$

**Theorem B** (Brezis-Merle [6]). *Consider the case of one equation and assume that  $(u_i)_i$  and  $(V_i)_i$  are two sequences of functions relatively to the previous problem (P) with,  $0 \leq V_i \leq b < +\infty$ , and,*

$$\int_{\Omega} e^{u_i} dy \leq C,$$

*then, for all compact set  $K$  of  $\Omega$ ,*

$$\sup_K u_i \leq c = c(b, C, K, \Omega).$$

Next, we call energy the following quantity:

$$E = \int_{\Omega} e^{u_i} dy.$$

The boundedness of the energy is a necessary condition to work on the problem (P) as showed in [6], by the following counterexample.

**Theorem C** (Brezis-Merle [6]). *Consider the case of one equation, then there are two sequences  $(u_i)_i$  and  $(V_i)_i$  of the problem (P) with,  $0 \leq V_i \leq b < +\infty$ , and,*

$$\int_{\Omega} e^{u_i} dy \leq C,$$

*and*

$$\sup_{\Omega} u_i \rightarrow +\infty.$$

Note that in [11], Dupaigne-Farina-Sirakov proved (by an existence result of Montenegro, see [14]) that the solutions of the above system when  $V$  and  $W$  are constants can be extremal and this condition imply the boundedness of the energy and directly the compactness. Note that in [10], if we assume (in particular) that  $\nabla \log V$  and  $\nabla \log W$  and  $V > a > 0$  or  $W > a' > 0$  and  $V, W$  are nonnegative and uniformly bounded then the energy is bounded and we have a compactness result.

Note that in the case of one equation, we can prove by using the Pohozaev identity that if  $+\infty > b \geq V \geq a > 0$ ,  $\nabla V$  is uniformly Lipschitzian that the energy is bounded when  $\Omega$  is starshaped. In [13] Ma-Wei, using the moving-plane method showed that this fact is true for all domain  $\Omega$  with the same assumptions on  $V$ . In [10] De Figueiredo-do O-Ruf extend this fact to a system by using the moving-plane method for a system.

Theorem C, shows that we have not a global compactness to the previous problem with one equation, perhaps we need more information on  $V$  to conclude to the boundedness of the solutions. When  $\nabla \log V$  is Lipschitz function, Chen-Li and Ma-Wei see [7] and [13], showed that we have a compactness on all the open set. The proof is via the moving plane-Method of Serrin and Gidas-Ni-Nirenberg. Note that in [10], we have the same result for this system when  $\nabla \log V$  and  $\nabla \log W$  are uniformly bounded. We will see below that for a system we also have a compactness result when  $V$  and  $W$  are Lipschitzian.

Now consider the case of one equation. In this case our equation have nice properties.

If we assume  $V$  with more regularity, we can have another type of estimates, a sup + inf type inequalities. It was proved by Shafrir see [16], that, if  $(u_i)_i, (V_i)_i$  are two sequences of functions solutions of the previous equation without assumption on the boundary and,  $0 < a \leq V_i \leq b < +\infty$ , then we have the following interior estimate:

$$C\left(\frac{a}{b}\right) \sup_K u_i + \inf_{\Omega} u_i \leq c = c(a, b, K, \Omega).$$

Now, if we suppose  $(V_i)_i$  uniformly Lipschitzian with  $A$  the Lipschitz constant, then,  $C(a/b) = 1$  and  $c = c(a, b, A, K, \Omega)$ , see [5].

Here we are interested by the case of a system of this type of equation. First, we give the behavior of the blow-up points on the boundary and in the second time we have a proof of compactness of the solutions to Gelfand-Liouville type system with Lipschitz condition.

Here, we write an extention of Brezis-Merle Problem (see [6]) is:

**Problem.** Suppose that  $V_i \rightarrow V$  and  $W_i \rightarrow W$  in  $C^0(\bar{\Omega})$ , with,  $0 \leq V_i \leq b_1$  and  $0 \leq W_i \leq b_2$  for some positive constants  $b_1, b_2$ . Also, we consider a sequence of solutions  $(u_i), (v_i)$  of  $(P)$  relatively to  $(V_i), (W_i)$  such that,

$$\int_{\Omega} e^{u_i} dx \leq C_1, \quad \int_{\Omega} e^{v_i} dx \leq C_2,$$

is it possible to have:

$$\|u_i\|_{L^\infty} \leq C_3 = C_3(b_1, b_2, C_1, C_2, \Omega)?$$

and,

$$\|v_i\|_{L^\infty} \leq C_4 = C_4(b_1, b_2, C_1, C_2, \Omega)?$$

In this paper we give a characterization of the behavior of the blow-up points on the boundary and also a proof of the compactness theorem when  $V_i$  and  $W_i$  are uniformly Lipschitzian. For the behavior of the blow-up points on the boundary, the following condition are enough,

$$0 \leq V_i \leq b_1, \quad 0 \leq W_i \leq b_2,$$

The conditions  $V_i \rightarrow V$  and  $W_i \rightarrow W$  in  $C^0(\bar{\Omega})$  are not necessary.

But for the proof of the compactness for the Gelfand-Liouville type system (Brezis-Merle type problem) we assume that:

$$\|\nabla V_i\|_{L^\infty} \leq A_1, \quad \|\nabla W_i\|_{L^\infty} \leq A_2.$$

Our main result are:

**Theorem 1.1.** *Assume that  $\max_{\Omega} u_i \rightarrow +\infty$  and  $\max_{\Omega} v_i \rightarrow +\infty$  Where  $(u_i)$  and  $(v_i)$  are solutions of the probleme (P) with:*

$$0 \leq V_i \leq b_1, \quad \text{and} \quad \int_{\Omega} e^{u_i} dx \leq C_1, \quad \forall i,$$

and,

$$0 \leq W_i \leq b_2, \quad \text{and} \quad \int_{\Omega} e^{v_i} dx \leq C_2, \quad \forall i,$$

then; after passing to a subsequence, there is a fncion  $u$ , there is a number  $N \in \mathbb{N}$  and  $N$  points  $x_1, x_2, \dots, x_N \in \partial\Omega$ , such that,

$$\int_{\partial\Omega} \partial_{\nu} u_i \varphi \rightarrow \int_{\partial\Omega} \partial_{\nu} u \varphi + \sum_{j=1}^N \alpha_j \varphi(x_j), \quad \alpha_j \geq 4\pi,$$

for any  $\varphi \in C^0(\partial\Omega)$ , and,

$$u_i \rightarrow u \quad \text{in} \quad C_{loc}^1(\bar{\Omega} - \{x_1, \dots, x_N\}).$$

$$\int_{\partial\Omega} \partial_{\nu} u_i \varphi \rightarrow \int_{\partial\Omega} \partial_{\nu} u \varphi + \sum_{j=1}^N \beta_j \varphi(x_j), \quad \beta_j \geq 4\pi,$$

for any  $\varphi \in C^0(\partial\Omega)$ , and,

$$v_i \rightarrow v \quad \text{in} \quad C_{loc}^1(\bar{\Omega} - \{x_1, \dots, x_N\}).$$

In the following theorem, we have a proof for the global a priori estimate which concern the problem (P).

**Theorem 1.2.** Assume that  $(u_i), (v_i)$  are solutions of  $(P)$  relatively to  $(V_i), (W_i)$  with the following conditions:

$$x_1 = 0 \in \partial\Omega,$$

and,

$$0 \leq V_i \leq b_1, \|\nabla V_i\|_{L^\infty} \leq A_1, \text{ and } \int_{\Omega} e^{u_i} \leq C_1,$$

$$0 \leq W_i \leq b_2, \|\nabla W_i\|_{L^\infty} \leq A_2, \text{ and } \int_{\Omega} e^{v_i} \leq C_2,$$

We have,

$$\|u_i\|_{L^\infty} \leq C_3(b_1, b_2, A_1, A_2, C_1, C_2, \Omega),$$

and,

$$\|v_i\|_{L^\infty} \leq C_4(b_1, b_2, A_1, A_2, C_1, C_2, \Omega),$$

## 2. PROOF OF THE THEOREMS

### Proof of theorem 1.1:

Since  $V_i e^{v_i}$  and  $W_i e^{u_i}$  are bounded in  $L^1(\Omega)$ , we can extract from those two sequences two subsequences which converge to two nonnegative measures  $\mu_1$  and  $\mu_2$ .

If  $\mu_1(x_0) < 4\pi$ , by a Brezis-Merle estimate for the first equation, we have  $e^{u_i} \in L^{1+\epsilon}$  around  $x_0$ , by the elliptic estimates, for the second equation, we have  $v_i \in W^{2,1+\epsilon} \subset L^\infty$  around  $x_0$ , and, returning to the first equation, we have  $u_i \in L^\infty$  around  $x_0$ .

If  $\mu_2(x_0) < 4\pi$ , then  $u_i$  and  $v_i$  are also locally bounded around  $x_0$ .

Thus, we take a look to the case when,  $\mu_1(x_0) \geq 4\pi$  and  $\mu_2(x_0) \geq 4\pi$ . By our hypothesis, those points  $x_0$  are finite.

We will see that inside  $\Omega$  no such points exist. By contradiction, assume that, we have  $\mu_1(x_0) \geq 4\pi$ . Let us consider a ball  $B_R(x_0)$  which contain only  $x_0$  as nonregular point. Thus, on  $\partial B_R(x_0)$ , the two sequence  $u_i$  and  $v_i$  are uniformly bounded. Let us consider:

$$\begin{cases} -\Delta z_i = V_i e^{v_i} & \text{in } B_R(x_0) \subset \mathbb{R}^2, \\ z_i = 0 & \text{in } \partial B_R(x_0). \end{cases}$$

By the maximum principle we have:

$$z_i \leq u_i$$

and  $z_i \rightarrow z$  almost everywhere on this ball, and thus,

$$\int e^{z_i} \leq \int e^{u_i} \leq C,$$

and,

$$\int e^z \leq C.$$

but,  $z$  is a solution to the following equation:

$$\begin{cases} -\Delta z = \mu_1 & \text{in } B_R(x_0) \subset \mathbb{R}^2, \\ z = 0 & \text{in } \partial B_R(x_0). \end{cases}$$

with,  $\mu_1 \geq 4\pi$  and thus,  $\mu_1 \geq 4\pi\delta_{x_0}$  and then, by the maximum principle:

$$z \geq -2 \log |x - x_0| + C$$

thus,

$$\int e^z = +\infty,$$

which is a contradiction. Thus, there is no nonregular points inside  $\Omega$

Thus, we consider the case where we have nonregular points on the boundary, we use two estimates:

$$\int_{\partial\Omega} \partial_\nu u_i d\sigma \leq C_1, \quad \int_{\partial\Omega} \partial_\nu v_i d\sigma \leq C_2,$$

and,

$$\|\nabla u_i\|_{L^q} \leq C_q, \quad \|\nabla v_i\|_{L^q} \leq C'_q, \quad \forall i \text{ and } 1 < q < 2.$$

We have the same computations, as in the case of one equation.

We consider a points  $x_0 \in \partial\Omega$  such that:

$$\mu_1(x_0) < 4\pi.$$

We consider a test function on the boundary  $\eta$  we extend  $\eta$  by a harmonic function on  $\Omega$ , we write the equation:

$$-\Delta((u_i - u)\eta) = (V_i e^{u_i} - V e^u)\eta + \langle \nabla(u_i - u) | \nabla \eta \rangle = f_i$$

with,

$$\int |f_i| \leq 4\pi - \epsilon + o(1) < 4\pi - 2\epsilon < 4\pi,$$

$$-\Delta((v_i - v)\eta) = (W_i e^{v_i} - W e^v)\eta + \langle \nabla(v_i - v) | \nabla \eta \rangle = g_i,$$

with,

$$\int |g_i| \leq 4\pi - \epsilon + o(1) < 4\pi - 2\epsilon < 4\pi,$$

By the Brezis-Merle estimate, we have uniformly,  $e^{u_i} \in L^{1+\epsilon}$  around  $x_0$ , by the elliptic estimates, for the second equation, we have  $v_i \in W^{2,1+\epsilon} \subset L^\infty$  around  $x_0$ , and , returning to the first equation, we have  $u_i \in L^\infty$  around  $x_0$ .

We have the same thing if we assume:

$$\mu_2(x_0) < 4\pi.$$

Thus, if  $\mu_1(x_0) < 4\pi$  or  $\mu_2(x_0) < 4\pi$ , we have for  $R > 0$  small enough:

$$(u_i, v_i) \in L^\infty(B_R(x_0) \cap \bar{\Omega}).$$

By our hypothesis the set of the points such that:

$$\mu_1(x_0) \geq 4\pi, \quad \mu_2(x_0) \geq 4\pi,$$

is finite, and, outside this set  $u_i$  and  $v_i$  are locally uniformly bounded. By the elliptic estimates, we have the  $C^1$  convergence to  $u$  and  $v$  on each compact set of  $\bar{\Omega} - \{x_1, \dots, x_N\}$ .

Proof of theorem 1.2:

Without loss of generality, we can assume that 0 is a blow-up point (either, we use a translation). Also, by a conformal transformation, we can assume that  $\Omega = B_1^+$ , the half ball, and  $\partial^+ B_1^+$  is the exterior part, a part which not contain 0 and on which  $u_i$  and  $v_i$  converge in the  $C^1$  norm to  $u$  and  $v$ . Let us consider  $B_\epsilon^+$ , the half ball with radius  $\epsilon > 0$ .

The Pohozaev identity gives :

$$\int_{B_\epsilon^+} \Delta u_i \langle x | \nabla v_i \rangle dx = - \int_{B_\epsilon^+} \Delta v_i \langle x | \nabla u_i \rangle dx + \int_{\partial^+ B_\epsilon^+} g(\partial_\nu u_i, \partial_\nu v_i) d\sigma, \quad (1)$$

Thus,

$$\int_{B_\epsilon^+} V_i e^{v_i} \langle x | \nabla v_i \rangle dx = - \int_{B_\epsilon^+} W_i e^{u_i} \langle x | \nabla u_i \rangle dx + \int_{\partial^+ B_\epsilon^+} g(\partial_\nu u_i, \partial_\nu v_i) d\sigma, \quad (2)$$

After integration by parts, we obtain:

$$\begin{aligned} & \int_{B_\epsilon^+} V_i e^{v_i} dx + \int_{B_\epsilon^+} \langle x | \nabla V_i \rangle e^{v_i} dx + \int_{\partial B_\epsilon^+} \langle \nu | \nabla V_i \rangle d\sigma + \\ & + \int_{B_\epsilon^+} W_i e^{u_i} dx + \int_{B_\epsilon^+} \langle x | \nabla W_i \rangle e^{u_i} dx + \int_{\partial B_\epsilon^+} \langle \nu | \nabla W_i \rangle d\sigma = \\ & = \int_{\partial^+ B_\epsilon^+} g(\partial_\nu u_i, \partial_\nu v_i) d\sigma, \end{aligned}$$

Also, for  $u$  and  $v$ , we have:

$$\begin{aligned} & \int_{B_\epsilon^+} V e^v dx + \int_{B_\epsilon^+} \langle x | \nabla V \rangle e^v dx + \int_{\partial B_\epsilon^+} \langle \nu | \nabla V \rangle d\sigma + \\ & + \int_{B_\epsilon^+} W e^u dx + \int_{B_\epsilon^+} \langle x | \nabla W \rangle e^u dx + \int_{\partial B_\epsilon^+} \langle \nu | \nabla W \rangle d\sigma = \\ & = \int_{\partial^+ B_\epsilon^+} g(\partial_\nu u, \partial_\nu v) d\sigma, \end{aligned}$$

If, we take the difference, we obtain:

$$(1 + o(\epsilon)) \left( \int_{B_\epsilon^+} V_i e^{v_i} dx - \int_{B_\epsilon^+} V e^v dx \right) +$$



$$\begin{aligned}
& + (1 + o(\epsilon)) \left( \int_{B_\epsilon^+} W_i e^{u_i} dx - \int_{B_\epsilon^+} W e^u dx \right) = \\
& = \alpha_1 + \beta_1 + o(\epsilon) + o(1) = o(1),
\end{aligned}$$

a contradiction.

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DEPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ PIERRE ET MARIE CURIE, 2 PLACE JUSSIEU, 75005, PARIS, FRANCE.

*E-mail address:* samybahoura@yahoo.fr